

Can the canonical quantization be accomplished within the intrinsic geometry?

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For particles constrained on a curved surface, how to perform quantization within Dirac's canonical quantization scheme is a long-standing problem. On one hand, Dirac stressed that the Cartesian coordinate system has fundamental importance in passing from the classical Hamiltonian to its quantum mechanical form while preserving the classical algebraic structure between positions, momenta and Hamiltonian to the extent possible. On the other, on the curved surface, we have no exact Cartesian coordinate system within intrinsic geometry. These two facts imply that the three-dimensional Euclidean space in which the curved surface is embedded must be invoked otherwise no proper canonical quantization is attainable. Since the minimum surfaces, catenoid and helicoid studied in this paper, have vanishing mean curvature, we explore whether the intrinsic geometry offers a proper framework in which the quantum theory can be established in a self-consistent way. Results show that it does for quantum motions on catenoid and it does not for that on helicoid, but neither is compatible with Schrödinger theory. In contrast, in three-dimensional Euclidean space, the geometric momentum and potential are then in agreement with those given by the Schrödinger theory.

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I. INTRODUCTION

Recently, a quantum motion on the two-dimensional curved surface Σ^2 has attracted increasing attention. [1–20] On one hand, if the quantum Hamiltonian is the same function of the canonical coordinates and momenta in the quantum theory as in the classical theory, the canonical quantization needs a well-defined Cartesian coordinate system. [21–24] On the other, for a curved manifold, not only the global Cartesian coordinate system does not exist but also the local one can be used approximately. In the differential geometry for the surface Σ^2 , a complete description of Σ^2 needs a three-dimensional flat space R^3 in which so-called second fundamental form can be then defined. [25] Combining these two observations, we are confident that a proper description for quantum motion on Σ^2 is not possible unless in the three-dimensional flat space R^3 . As a consequence, the geometric momentum is identified and introduced as a proper description of the momentum for a particle constrained on the curved surface. [10, 11] In contrast, the conventional formalism of quantum mechanics is established within the framework of the intrinsic geometry. In this paper, we utilize two minimum surfaces, catenoid and helicoid, to further explore the relationship between geometry and quantization.

Let us first recall elementary differential geometry for the two-dimensional curved surface Σ^2 that is embedded in the three-dimensional Cartesian space R^3 . The surface Σ^2 is parameterized by $q^\mu \equiv (u, v)$ with μ running from 1 to 2, we have in three-dimensional Cartesian coordinate system the positions $\mathbf{r} \equiv (x(u, v), y(u, v), z(u, v))$, and normal vector $\mathbf{n} = (n_x, n_y, n_z) \equiv \mathbf{r}_u \times \mathbf{r}_v / |\mathbf{r}_u \times \mathbf{r}_v|$ where $\mathbf{r}_\mu \equiv \partial \mathbf{r} / \partial x^\mu$, and $\mathbf{r}^\mu = g^{\mu\nu} \mathbf{r}_\nu = g^{\mu\nu} \partial \mathbf{r} / \partial x^\nu$. In whole of this paper, the Einstein summation convention that repeated indices are implicitly summed over. At this point \mathbf{r} , we have two geometric invariants, the mean curvature vector $M\mathbf{n}$ and the gaussian curvature K which characterizes, respectively, the extrinsic and the intrinsic curvature.

Next, let us recall a fact on the relation between the three-dimensional Cartesian space R^3 and an effective quantum theory for the surface Σ^2 . The Schrödinger equation is first formulated in R^3 , actually in a curved shell of an equal and finite thickness δ whose intermediate surface coincides with the prescribed one Σ^2 (or equivalently, the particle moves within the thin layer of the same width δ due to a confining potential around the surface), and an effective Schrödinger equation on the curved surface Σ^2 is second derived by taking the squeezing limit $\delta \rightarrow 0$ to confine the particle to the Σ^2 . [6–11, 13, 14, 17–20] It leads to unambiguous forms for the geometric momentum \mathbf{p} [10] and geometric kinetic energy that contains the geometric potential V_g [6, 7], which are given by,

$$\mathbf{p} = -i\hbar(\mathbf{r}^\mu \partial_\mu + M\mathbf{n}), \quad (1)$$

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$$T \equiv -\frac{\hbar^2}{2m}\Delta + V_g, V_g = -\frac{\hbar^2}{2m}(M^2 - K), \quad (2)$$

where $\mathbf{r}^\mu \partial_\mu$ is sometimes denoted as ∇_2 that is the gradient operator on a two-dimensional surface. [25] Both the kinetic energy (2) and momentum (1) are geometric invariants.

The presence of the geometric potential V_g enriches our understanding of the quantization procedure. For a quantum system that has a classical analogue, we can no longer assume in general that, the quantum Hamiltonian is the same function of the canonical coordinates and momenta in the quantum theory as in the classical theory, even in the Cartesian coordinate system. Moreover, the consistence between *fundamental quantum conditions* and the equations of motion, i.e., the Ehrenfest theorem $[f, H]/(i\hbar) = \{f, H\}_D$ for $f = x_i$ and p_i , turns out to be problematic, where $\{A, B\}_D$ is in general the Dirac bracket between two variables A and B for a system that has second-class constraints, and reduces to the usual Poisson bracket when the system is constraint-free. By the *fundamental quantum conditions*, we mean that the commutation relations $[A, B]$ between the coordinates x_μ ($\mu = 1, 2$) and momenta p_μ , or x_i ($i = 1, 2, 3$) and p_i , satisfy $[A, B]/(i\hbar) = \{A, B\}_D$, according to Dirac. [21] The current procedure [26, 27] quantizes both the generalized coordinates/momenta (x_μ, p_ν) and the Cartesian ones (x_i, p_j) on an equal footing. [21, 28] This procedure differs from the underlying idea in the thin layer method in the Schrödinger equation approach as outlined above, where $q^\mu \equiv (u, v)$ are purely parameters while performing quantization of the Cartesian coordinates and momenta (\mathbf{x}, \mathbf{p}) .

Now, we propose a *strengthened* version of the Dirac's *canonical quantization* (SCQ) scheme as what follows. For a quantum system that has a classical analogue, there are two categories of the *fundamental quantum conditions*. The original ones belong to the first, which is between the coordinates x_μ and momenta p_μ or x_i and p_i . The second category is the commutation relations $[f, H]$, where H is the Hamiltonian and $f = x_\mu$ and p_μ or $f = x_i$ and p_i , which must also satisfy correspondence $[f, H]/(i\hbar) = \{f, H\}_D$. In other words, the SCQ hypothesizes a *simultaneous* quantization for positions, momenta, and Hamiltonian while preserving the formal algebraic structure between them to the extent possible. It is evident that, once the constraints are free, the second category of the *fundamental quantum conditions* is trivially satisfied because they are nothing but the Ehrenfest theorem, as long as the Cartesian coordinate system is used.

Notice that the Dirac's canonical quantization scheme should be examined on the case-by-case basis. [24] For particles constrained on the minimum surfaces with $M = 0$, momentum (1) and kinetic energy (2) assume their dependence on purely intrinsic geometric quantity. Whether the intrinsic geometry offer a proper framework for the canonical quantization scheme is then an interesting issue. Recently, quantum motion on two minimum surfaces, catenoid [18] and helicoid, [17] are investigated. In the present paper we take also these two surfaces to see whether the quantum theory can be established satisfactorily. Results turn out to be compatible with Dirac's remark that only the Cartesian coordinate system is physically permissible while the intrinsic geometry suffers from various problems. The organization of the paper is as follows. In sections II and III, we study catenoid, respectively, within purely intrinsic geometry and as a submanifold in R^3 . In sections IV and V, we study helicoid in similar manner. Section VI briefly remarks and concludes this study. In the present study, no external potential field presents without loss of generality.

II. DIRAC'S THEORY OF SECOND-CLASS CONSTRAINTS FOR A CATENOID WITHIN INTRINSIC GEOMETRY

The catenoid is with two local coordinates $\theta \in [0, 2\pi), \rho \in R$,

$$\mathbf{r} = (r \cosh \frac{\rho}{r} \cos \theta, r \cosh \frac{\rho}{r} \sin \theta, \rho), \quad r > 0, \quad (3)$$

where r is the constraint parameter that will be set as $r = a \neq 0$. In this section, we will first give the classical mechanics for motion on the catenoid within Dirac's theory of second-class constraints, and then turn into quantum mechanics. In classical mechanics, the theory appears nothing surprising, but after transition to quantum mechanics, it breaks agreement with the Schrödinger theory.

A. Classical mechanical treatment

The Lagrangian L in the local coordinate system is,

$$L = \frac{1}{2}m \left(\frac{1}{r^2} \left(r \cosh \frac{\rho}{r} - \rho \sinh \frac{\rho}{r} \right)^2 \dot{r}^2 + r^2 \cosh^2 \frac{\rho}{r} \dot{\theta}^2 + \frac{2}{r} \sinh \frac{\rho}{r} \left(r \cosh \frac{\rho}{r} - \rho \sinh \frac{\rho}{r} \right) \dot{r} \dot{\rho} + \cosh^2 \frac{\rho}{r} \dot{\rho}^2 \right) - \lambda(r - a), \quad (4)$$

where λ is the Lagrangian multiplier enforcing the constrained of motion on the surface, nevertheless, we treat the quantity λ as an additional dynamical variable. The Lagrangian is singular because it does not contain the "velocity" $\dot{\lambda}$. Hence we need Dirac's theory, which gives the canonical momenta conjugate to r, θ, ρ and λ in the following,

$$p_r = \frac{\partial L}{\partial \dot{r}} = m \left(\frac{1}{r^2} \left(r \cosh \frac{\rho}{r} - \rho \sinh \frac{\rho}{r} \right)^2 \dot{r} + \frac{1}{r} \sinh \frac{\rho}{r} \left(r \cosh \frac{\rho}{r} - \rho \sinh \frac{\rho}{r} \right) \dot{\rho} \right), \quad (5)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \cosh^2 \frac{\rho}{r} \dot{\theta}, \quad (6)$$

$$p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m \left(\frac{1}{r} \sinh \frac{\rho}{r} \left(r \cosh \frac{\rho}{r} - \rho \sinh \frac{\rho}{r} \right) \dot{r} + \cosh^2 \frac{\rho}{r} \dot{\rho} \right), \quad (7)$$

$$p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0. \quad (8)$$

Eq. (8) represents the primary constraint:

$$\varphi_1 \equiv p_\lambda \approx 0, \quad (9)$$

hereafter symbol " \approx " implies a weak equality [28]. After all calculations are finished, the weak equality takes back the strong one. By the Legendre transformation, the primary Hamiltonian H_p is, [28]

$$H_p = \frac{r^2 \cosh^2 \frac{\rho}{r}}{2m \left(r \cosh \frac{\rho}{r} - \rho \sinh \frac{\rho}{r} \right)^2} p_r^2 - \frac{r \sinh \frac{\rho}{r}}{m \left(r \cosh \frac{\rho}{r} - \rho \sinh \frac{\rho}{r} \right)^2} p_r p_\rho + \frac{1}{2mr^2 \cosh^2 \frac{\rho}{r}} p_\theta^2 + \frac{1}{2m} p_\rho^2 + \lambda(r - a) + u p_\lambda, \quad (10)$$

where u is also a Lagrangian multiplier guaranteeing that this Hamiltonian is defined on the symplectic manifold. The secondary constraints (not confusing with second-class constraints) are generated successively, then determined by the conservation condition [28],

$$\varphi_{i+1} \equiv \{\varphi_i, H_p\} \approx 0, \quad (i = 1, 2, \dots), \quad (11)$$

where $\{f, g\}$ is the Poisson bracket with $q_1 = r, q_2 = \theta, q_3 = \rho$, and $p_1 = p_r, p_2 = p_\theta, p_3 = p_\rho$,

$$\{f, g\} \equiv \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} + \frac{\partial f}{\partial \lambda} \frac{\partial g}{\partial p_\lambda} - \left(\frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} + \frac{\partial f}{\partial p_\lambda} \frac{\partial g}{\partial \lambda} \right). \quad (12)$$

The complete set of the secondary constraints is,

$$\varphi_2 \equiv \{\varphi_1, H_p\} = a - r \approx 0, \quad (13)$$

$$\varphi_3 \equiv \{\varphi_2, H_p\} = r \left(-\frac{r \cosh^2 \frac{\rho}{r}}{m \left(r \cosh \frac{\rho}{r} - \rho \sinh \frac{\rho}{r} \right)^2} p_r + \frac{\sinh \frac{\rho}{r}}{m \left(r \cosh \frac{\rho}{r} - \rho \sinh \frac{\rho}{r} \right)} p_\rho \right) \approx 0, \quad (14)$$

$$\begin{aligned} \varphi_4 \equiv \{\varphi_3, H_p\} &= \frac{r^2 \cosh^2 \frac{\rho}{r}}{m \left(r \cosh \frac{\rho}{r} - \rho \sinh \frac{\rho}{r} \right)^2} \lambda + \frac{r^2 \cosh \frac{\rho}{r} (r p_\rho - 2 \rho p_r + r \cosh \frac{2\rho}{r} p_\rho - (r p_r + \rho p_\rho) \sinh \frac{2\rho}{r})^2}{4m^2 \left(r \cosh \frac{\rho}{r} - \rho \sinh \frac{\rho}{r} \right)^5} \\ &\quad - \frac{p_\theta^2}{m^2 r^2 \left(r \cosh \frac{\rho}{r} - \rho \sinh \frac{\rho}{r} \right) \cosh^3 \frac{\rho}{r}} \approx 0. \end{aligned} \quad (15)$$

Eqs. (13) shows that on the surface of catenoid $r = a$, there is no motion along the normal direction, while Eqs. (15) determines the dynamical variable λ , and by the conservation condition of the secondary constraint φ_4 (15), we can determine the Lagrangian multipliers u .

The Dirac bracket instead of the Poisson bracket for two variables A and B is defined by,

$$\{A, B\}_D \equiv \{A, B\} - \{A, \varphi_\xi\} C_{\xi\zeta}^{-1} \{\varphi_\zeta, B\}, \quad (16)$$

where the 4×4 matrix $C \equiv \{C_{\xi\zeta}\}$ whose elements are defined by $C_{\xi\zeta} \equiv \{\varphi_\xi, \varphi_\zeta\}$ with $\xi, \zeta = 1, 2, 3, 4$ from Eqs. (9) and (13)-(15). The inverse matrix C^{-1} is,

$$C^{-1} = \begin{pmatrix} 0 & C_{12}^{-1} & C_{13}^{-1} & C_{14}^{-1} \\ -C_{12}^{-1} & 0 & C_{23}^{-1} & 0 \\ -C_{13}^{-1} & -C_{23}^{-1} & 0 & 0 \\ -C_{14}^{-1} & 0 & 0 & 0 \end{pmatrix}, \quad (17)$$

where

$$C_{12}^{-1} = \frac{1}{ma^6 \cosh^4 \frac{\rho}{a}} \left((-5p_\theta^2 + a^2 p_\rho^2) \rho^2 \frac{1}{\cosh^4 \frac{\rho}{a}} + a (-p_\theta^2 + a^2 p_\rho^2) \left(2a - \rho \tanh \frac{\rho}{a} \right) \right. \\ \left. + \frac{1}{\cosh^2 \frac{\rho}{a}} \left(-a^4 p_\rho^2 + p_\theta^2 (5a^2 + 4\rho^2) + 2a (-5p_\theta^2 + a^2 p_\rho^2) \rho \tanh \frac{\rho}{a} \right) \right), \quad (18)$$

$$C_{13}^{-1} = -\frac{p_\rho}{a^3 \cosh^5 \frac{\rho}{a}} \left(a \cosh \frac{\rho}{a} - \rho \sinh \frac{\rho}{a} \right) \left(2\rho + a \sinh \frac{2\rho}{a} \right), \quad (19)$$

$$C_{14}^{-1} = -C_{23}^{-1} = \frac{m}{a^2} \left(a - \rho \tanh \frac{\rho}{a} \right)^2. \quad (20)$$

Because of no motion along the normal direction, we need not analyze the dynamics and kinematics of the normal direction. Thus, the generalized positions $q^\mu (= \theta, \rho)$ and momenta p_ν satisfy the following Dirac brackets,

$$\{q^\mu, q^\nu\}_D = 0, \{p_\mu, p_\nu\}_D = 0, \{q^\mu, p_\nu\}_D = \delta_\nu^\mu. \quad (21)$$

By use of the equation of motion,

$$\dot{f} = \{f, H\}_D. \quad (22)$$

we obtain those for the positions θ, ρ and the momenta p_θ, p_ρ , respectively,

$$\dot{\theta} \equiv \{\theta, H\}_D = \frac{p_\theta}{ma^2 \cosh^2 \frac{\rho}{a}}, \quad \dot{\rho} \equiv \{\rho, H\}_D = \frac{p_\rho}{m \cosh^2 \frac{\rho}{a}}, \quad (23)$$

$$\dot{p}_\theta \equiv \{p_\theta, H\}_D = 0, \quad \dot{p}_\rho \equiv \{p_\rho, H\}_D = \frac{2}{a} \tanh \frac{\rho}{a} H. \quad (24)$$

In these calculations (22) and (24), we used the usual form of Hamiltonian, $H_p \rightarrow H$,

$$H = \frac{1}{2ma^2 \cosh^2 \frac{\rho}{a}} (p_\theta^2 + a^2 p_\rho^2). \quad (25)$$

So far, the classical mechanics for the motion on the catenoid is complete and coherent in itself.

B. Quantum mechanical treatment

In quantum mechanics, we assume that the Hamiltonian takes the following general form,

$$H = -\frac{\hbar^2}{2m} [\nabla^2 + (\alpha M^2 - \beta K)] \\ = -\frac{\hbar^2}{2m} \left[\frac{1}{a^2 \cosh^2 \frac{\rho}{a}} \left(\frac{\partial^2}{\partial \theta^2} + a^2 \frac{\partial^2}{\partial \rho^2} \right) + \beta \frac{1}{a^2 \cosh^4 \frac{\rho}{a}} \right], \quad (26)$$

where,

$$M = 0, \quad K = -\frac{1}{a^2 \cosh^4 \frac{\rho}{a}}. \quad (27)$$

We are ready to construct commutator $[A, B]$ of two variables A and B in quantum mechanics, which can be straightforward realized by a direct correspondence of the Dirac's brackets as $[A, B]/i\hbar \rightarrow \{A, B\}_D$. From the Dirac's brackets (21), the first category of the fundamental commutators between operators q^μ and p_ν are given by,

$$[q^\mu, q^\nu] = 0, [p_\mu, p_\nu] = 0, [q^\mu, p_\nu] = i\hbar\delta_\nu^\mu. \quad (28)$$

Similarly, we have the second category of fundamental commutators between q^μ and H from Eq. (23),

$$[\theta, H] = \frac{i\hbar}{ma^2 \cosh^2 \frac{\rho}{a}} p_\theta, \quad (29)$$

$$[\rho, H] = \frac{i\hbar}{m} \frac{1}{2} \left(\frac{1}{\cosh^2 \frac{\rho}{a}} p_\rho + p_\rho \frac{1}{\cosh^2 \frac{\rho}{a}} \right). \quad (30)$$

On the other, the quantum commutators (29) and (30) from Hamiltonian (26) give a definite and satisfactory form for the operator p_θ and p_ρ , [26]

$$p_\theta = -i\hbar \frac{\partial}{\partial \theta}, \quad (31)$$

$$p_\rho = -i\hbar \left(\frac{\partial}{\partial \rho} + \frac{1}{a} \tanh \frac{\rho}{a} \right). \quad (32)$$

Using these operators, we can directly calculate two quantum commutators $[p_\theta, H]$ and $[p_\rho, H]$ with quantum Hamiltonian (26), and the results are, respectively,

$$[p_\theta, H] = 0, \quad (33)$$

$$[p_\rho, H] = i\hbar \{p_\rho, H\}_D - \beta \frac{i\hbar^3}{ma^3 \cosh^5 \frac{\rho}{a}} \sinh \frac{\rho}{a}. \quad (34)$$

The first equation (33) is satisfactory, whereas the second one (34) is problematic. With a choice of the parameter $\beta \neq 0$, there is a manifest breakdown of the formal algebraic structure. With a choice of the parameter $\beta = 0$, the SCQ becomes self-consistent, but contradicts with the Schrödinger theory that predicts $\beta = 1$.

C. Remarks

From the studies in this section, we see that the SCQ for quantum motion on the catenoid can be consistently established but is contrary to the Schrödinger theory. We therefore need to invoke an extrinsic examination of the same problem, as will be done in next section.

III. DIRAC'S THEORY OF SECOND-CLASS CONSTRAINTS FOR A CATENOID AS A SUBMANIFOLD

The surface equation of the catenoid (3) in Cartesian coordinates (x, y, z) is given by,

$$f(\mathbf{x}) \equiv x^2 + y^2 - a^2 \cosh^2 \frac{z}{a} = 0. \quad (35)$$

In this section, we will also first give the classical mechanics for motion on the catenoid within Dirac's theory of second-class constraints, and then turn into quantum mechanics. The obtained momentum and Hamiltonian are all compatible with the those given by Schrödinger theory.

A. Classical mechanical treatment

The Lagrangian L in the Cartesian coordinate system is,

$$L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \lambda f(\mathbf{x}). \quad (36)$$

The generalized momentum p_i ($i = x, y, z$) and p_λ canonically conjugate to variables x_i ($x_1 = x, x_2 = y, x_3 = z$) and λ , are given by, respectively,

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i, (i = 1, 2, 3), \quad (37)$$

$$p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0. \quad (38)$$

Eq. (38) represents the primary constraint,

$$\varphi_1 \equiv p_\lambda \approx 0. \quad (39)$$

By the Legendre transformation, the primary Hamiltonian H_p is,

$$H_p = \frac{1}{2m} p_i^2 + \lambda f(\mathbf{x}) + \lambda p_\lambda. \quad (40)$$

The secondary constraints are determined by successive use of the Poisson brackets,

$$\varphi_2 \equiv \{\varphi_1, H_p\} = -(x^2 + y^2 - a^2 \cosh^2 \frac{z}{a}) \approx 0, \quad (41)$$

$$\varphi_3 \equiv \{\varphi_2, H_p\} = \frac{-2(p_x x + p_y y) + a p_z \sinh \frac{2z}{a}}{m} \approx 0, \quad (42)$$

$$\varphi_4 \equiv \{\varphi_3, H_p\} = \frac{\lambda(-a^2 + 8(x^2 + y^2) + a^2 \cosh \frac{4z}{a})}{2m} - \frac{2(p_x^2 + p_y^2 - p_z^2 \cosh \frac{2z}{a})}{m^2} \approx 0. \quad (43)$$

Similarly, the Dirac bracket between two variables A and B is defined by,

$$\{A, B\}_D = \{A, B\} - \{A, \varphi_\xi\} D_{\xi\zeta}^{-1} \{\varphi_\zeta, B\}, \quad (44)$$

where the 4×4 matrix $D \equiv \{D_{\xi\zeta}\}$ whose elements are defined by $D_{\xi\zeta} \equiv \{\varphi_\xi, \varphi_\zeta\}$ with $\xi, \zeta = 1, 2, 3, 4$ from Eqs. (39) and (41)-(43). The inverse matrix D^{-1} is easily carried out,

$$D^{-1} = \begin{pmatrix} 0 & D_{12}^{-1} & D_{13}^{-1} & D_{14}^{-1} \\ -D_{12}^{-1} & 0 & D_{23}^{-1} & 0 \\ -D_{13}^{-1} & -D_{23}^{-1} & 0 & 0 \\ -D_{14}^{-1} & 0 & 0 & 0 \end{pmatrix}, \quad (45)$$

where,

$$D_{12}^{-1} = \frac{(12(p_x^2 + p_y^2) + 5p_z^2 - 4(p_x^2 + p_y^2 + 2p_z^2) \cosh \frac{2z}{a} + 3p_z^2 \cosh \frac{4z}{a})}{8a^4 m \cosh^8 \frac{z}{a}}, \quad (46)$$

$$D_{13}^{-1} = -\frac{\tanh \frac{z}{a}}{a^3 \cosh^4 \frac{z}{a}} p_z, \quad (47)$$

$$D_{14}^{-1} = -D_{23}^{-1} = \frac{m}{4a^2 \cosh^4 \frac{z}{a}}. \quad (48)$$

Then primary Hamiltonian H_p assumes its usual one: $H_p \rightarrow H$,

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m}. \quad (49)$$

All fundamental Dirac's brackets are as follows,

$$\{x_i, x_j\}_D = 0 \quad (50)$$

$$\{x_i, p_j\}_D = \delta_{ij} - \frac{1}{a^2 \cosh^4 \frac{z}{a}} \kappa_i \kappa_j, \quad (51)$$

$$\{p_i, p_j\}_D = -\frac{1}{a^2 \cosh^4 \frac{z}{a}} \left[\kappa_i p_j \left(\delta_{1j} + \delta_{2j} - \delta_{3j} \cosh \frac{2z}{a} \right) - \kappa_j p_i \left(\delta_{1i} + \delta_{2i} - \delta_{3i} \cosh \frac{2z}{a} \right) \right], \quad (52)$$

$$\{x_i, H\}_D = \frac{p_i}{m} = \dot{x}_i, \quad (53)$$

$$\{p_i, H\}_D = -\frac{1}{ma^2 \cosh^4 \frac{z}{a}} \kappa_i (p_x^2 + p_y^2 + p_z^2) + \frac{2}{ma^2 \cosh^2 \frac{z}{a}} \kappa_i p_z^2 = \dot{p}_i, \quad (54)$$

where $\kappa_i = x\delta_{i1} + y\delta_{i2} - a\delta_{i3} \sinh(z/a) \cosh(z/a)$.

B. Quantum mechanical treatment

Now let us turn to quantum mechanics. The first category of the fundamental commutators between operators x_i and p_i are, by quantization of (50)-(52),

$$[x_i, x_j] = 0, \quad [x_i, p_j] = i\hbar \left(\delta_{ij} - \frac{1}{a^2 \cosh^4 \frac{z}{a}} \kappa_i \kappa_j \right), \quad (55)$$

$$[p_i, p_j] = -\frac{i\hbar}{a^2 \cosh^4 \frac{z}{a}} \left[\kappa_i \left(\delta_{1j} + \delta_{2j} - \delta_{3j} \cosh \frac{2z}{a} \right) p_j - \kappa_j \left(\delta_{1i} + \delta_{2i} - \delta_{3i} \cosh \frac{2z}{a} \right) p_i \right], \quad (56)$$

There is a family of the momenta p_i that are solutions to the Eq. (56), as explicitly shown in [13]. With these momenta p_i , we completely do not know the correct form of the quantum Hamiltonian, as suggested by Eq. (49). It is therefore understandable that the quantum Hamiltonian would contain arbitrary parameters.

However, with the help of the second category of the fundamental commutators as $[x_i, H]$ and $[p_i, H]$, we immediately find that the momentum from following commutator,

$$[x_i, H] = i\hbar \frac{p_i}{m}. \quad (57)$$

They are, respectively, irrespective of the form of the geometric momentum,

$$p_x = -i\hbar \frac{1}{a \cosh \frac{\rho}{a}} \left(-\sin \theta \frac{\partial}{\partial \theta} + a \tanh \frac{\rho}{a} \cos \theta \frac{\partial}{\partial \rho} \right), \quad (58)$$

$$p_y = -i\hbar \frac{1}{a \cosh \frac{\rho}{a}} \left(\cos \theta \frac{\partial}{\partial \theta} + a \tanh \frac{\rho}{a} \sin \theta \frac{\partial}{\partial \rho} \right), \quad (59)$$

$$p_z = -i\hbar \frac{1}{\cosh^2 \frac{\rho}{a}} \frac{\partial}{\partial \rho}. \quad (60)$$

They are nothing but the geometric momentum (1) on the catenoid.

As to the form of quantum Hamiltonian, we also assume the general form (26). For the quantum commutators of the operators p_x , p_y and H , we must have from (54),

$$[p_i, H] = i\hbar \frac{1}{2} (F_i + F_i^\dagger) \quad (61)$$

where F^\dagger denotes the Hermitian conjugate of operator F , and

$$F_i = -\frac{2}{a^2 \cosh^4 \frac{z}{a}} \kappa_i H + \frac{2}{ma^2 \cosh^2 \frac{z}{a}} \kappa_i p_z^2, \quad \kappa_i = x_i - a \delta_{3i} \sinh\left(\frac{z}{a}\right) \cosh\left(\frac{z}{a}\right). \quad (62)$$

We can easily show that the geometric potential with $\beta = 1$ is compatible with the SCQ. For instance, we have,

$$\begin{aligned} [p_i, H] = & -\frac{i\hbar}{m} \left\{ mH \frac{1}{a^2 \cosh^4 \frac{z}{a}} \kappa_i + m \frac{1}{a^2 \cosh^4 \frac{z}{a}} \kappa_i H \right. \\ & - \frac{1}{2} \frac{1}{4} \left[\frac{2}{ma^2 \cosh^2 \frac{z}{a}} \kappa_i p_z^2 + p_z^2 \frac{2}{ma^2 \cosh^2 \frac{z}{a}} \kappa_i \right] \\ & \left. - \frac{1}{2} \frac{3}{4} \left[\frac{2}{ma^2 \cosh^2 \frac{z}{a}} p_z^2 \kappa_i + \kappa_i p_z^2 \frac{2}{ma^2 \cosh^2 \frac{z}{a}} \right] \right\}. \end{aligned} \quad (63)$$

Unfortunately, this choice is not unique, and we have also,

$$\begin{aligned} [p_z, H] = & \frac{i\hbar}{m} \left\{ mH \frac{\tanh \frac{z}{a}}{a \cosh^2 \frac{z}{a}} + m \frac{\tanh \frac{z}{a}}{a \cosh^2 \frac{z}{a}} H + \frac{3}{4} \left[\frac{\tanh \frac{z}{a}}{ma} p_z^2 + p_z^2 \frac{\tanh \frac{z}{a}}{ma} \right] \right. \\ & \left. + \frac{1}{4} \left[\frac{\tanh \frac{z}{a}}{ma \cosh^2 \frac{z}{a}} p_z^2 \cosh^2 \frac{z}{a} + \cosh^2 \frac{z}{a} p_z^2 \frac{\tanh \frac{z}{a}}{ma \cosh^2 \frac{z}{a}} \right] \right\}. \end{aligned} \quad (64)$$

From Eqs. (63) and (64), we can at least conclude that the geometric potential given by Dirac's canonical quantization is compatible with Schrödinger theory.

C. Remarks

An examination of the motion on catenoid as a submanifold problem in Dirac's theory of second-class constraints ensures a highly self-consistent description. This formalism is also compatible with Schrödinger one.

IV. DIRAC'S THEORY OF SECOND-CLASS CONSTRAINTS FOR A HELICOID WITHIN INTRINSIC GEOMETRY

The helicoid is with two local coordinates $u \in (-\infty, +\infty), v \in (-\infty, +\infty)$,

$$\mathbf{r} = (u \cos v, u \sin v, rv), \quad (65)$$

In this section, we will first give the classical mechanics for motion on the helicoid within Dirac's theory of second-class constraints, and then turn into quantum mechanics. In classical mechanics, the theory appears nothing surprising, but after transition to quantum mechanics, it becomes contradictory to itself.

A. Classical mechanical treatment

The Lagrangian L in the local coordinate system is,

$$L = \frac{1}{2}m (\dot{v}^2 (r^2 + u^2) + 2r\dot{r}v\dot{v} + \dot{r}^2 v^2 + \dot{u}^2) - \lambda(r - a), \quad (66)$$

where λ is the Lagrangian multiplier enforcing the constrained of motion on the surface, nevertheless, we treat the quantity λ as an additional dynamical variable. The Lagrangian is singular because it does not contain the "velocity" λ . Hence we need Dirac's theory, which gives the canonical momenta conjugate to r, u, v and λ in the following,

$$p_r = \frac{\partial L}{\partial \dot{r}} = mv(r\dot{v} + v\dot{r}), \quad (67)$$

$$p_u = \frac{\partial L}{\partial \dot{u}} = m\dot{u}, \quad (68)$$

$$p_v = \frac{\partial L}{\partial \dot{v}} = m(r^2\dot{v} + rv\dot{r} + u^2\dot{v}), \quad (69)$$

$$p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0. \quad (70)$$

Eq. (70) represents the primary constraint:

$$\varphi_1 \equiv p_\lambda \approx 0, \quad (71)$$

The primary Hamiltonian H_p is [28],

$$H_p = \frac{(r^2 + u^2)p_r^2 - 2rvp_r p_v + v^2(u^2 p_u^2 + p_v^2)}{2mu^2 v^2} + \lambda(r - a) + up_\lambda, \quad (72)$$

where u is also a Lagrangian multiplier guaranteeing that this Hamiltonian is defined on the symplectic manifold. The Poisson bracket of $\{f, g\}$ is with $q_1 = r, q_2 = u, q_3 = v$, and $p_1 = p_r, p_2 = p_u, p_3 = p_v$,

$$\{f, g\} \equiv \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} + \frac{\partial f}{\partial \lambda} \frac{\partial g}{\partial p_\lambda} - (\frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} + \frac{\partial f}{\partial p_\lambda} \frac{\partial g}{\partial \lambda}). \quad (73)$$

The complete set of the secondary constraints is,

$$\varphi_2 \equiv \{\varphi_1, H_p\} = a - r \approx 0, \quad (74)$$

$$\varphi_3 \equiv \{\varphi_2, H_p\} = \frac{p_v r v - p_r (r^2 + u^2)}{mu^2 v^2} \approx 0, \quad (75)$$

$$\varphi_4 \equiv \{\varphi_3, H_p\} = \frac{(r^2 + u^2)}{mu^2 v^2} \lambda + \frac{2(p_v v - p_r r)(p_r (r^2 + u^2) - p_u r u v^2 - p_v r v)}{m^2 u^4 v^4} \approx 0. \quad (76)$$

Eqs. (74) shows that on the surface of helicoid $r = a$, while Eqs. (76) determines the dynamical variable λ , and by the conservation condition of the secondary constraint φ_4 (76), we can determine the Lagrangian multipliers u .

The Dirac bracket instead of the Poisson bracket for two variables A and B is defined by,

$$\{A, B\}_D \equiv \{A, B\} - \{A, \varphi_\xi\} C_{\xi\zeta}^{-1} \{\varphi_\zeta, B\}, \quad (77)$$

where the 4×4 matrix $C \equiv \{C_{\xi\zeta}\}$ whose elements are defined by $C_{\xi\zeta} \equiv \{\varphi_\xi, \varphi_\zeta\}$ with $\xi, \zeta = 1, 2, 3, 4$ from Eqs. (71) and (74)-(76). The inverse matrix C^{-1} is,

$$C^{-1} = \begin{pmatrix} 0 & C_{12}^{-1} & C_{13}^{-1} & C_{14}^{-1} \\ -C_{12}^{-1} & 0 & C_{23}^{-1} & 0 \\ -C_{13}^{-1} & -C_{23}^{-1} & 0 & 0 \\ -C_{14}^{-1} & 0 & 0 & 0 \end{pmatrix}, \quad (78)$$

where

$$C_{12}^{-1} = \frac{2p_v uv (3a^4 p_u + 2a^2 u(p_u u + p_v v) - p_u u^4)}{m(a^2 + u^2)^4}, \quad (79)$$

$$C_{13}^{-1} = \frac{2uv(a^2 p_u v + p_v u)}{(a^2 + u^2)^2}, \quad (80)$$

$$C_{14}^{-1} = -C_{23}^{-1} = \frac{mu^2 v^2}{a^2 + u^2}. \quad (81)$$

Thus, the generalized positions q^μ ($= u, v$) and momenta p_ν satisfy the following Dirac brackets,

$$\{q^\mu, q^\nu\}_D = 0, \{p_\mu, p_\nu\}_D = 0, \{q^\mu, p_\nu\}_D = \delta_\nu^\mu. \quad (82)$$

By use of the equation of motion,

$$\dot{f} = \{f, H\}_D. \quad (83)$$

we obtain those for the positions u, v and the momenta p_u, p_v , respectively,

$$\dot{u} \equiv \{u, H\}_D = \frac{p_u}{m}, \quad \dot{v} \equiv \{v, H\}_D = \frac{p_v}{m(u^2 + a^2)}, \quad (84)$$

$$\dot{p}_u \equiv \{p_u, H\}_D = \frac{up_v^2}{m(a^2 + u^2)^2}, \quad \dot{p}_v \equiv \{p_v, H\}_D = 0. \quad (85)$$

In these calculations (84) and (85), we in fact need only the usual form of Hamiltonian, $H_p \rightarrow H$,

$$H = \frac{1}{2m} \left(p_u^2 + \frac{1}{(a^2 + u^2)} p_v^2 \right). \quad (86)$$

So far, the classical mechanics for the motion on the helicoid is complete and coherent in itself.

B. Quantum mechanical treatment

In quantum mechanics, we assume that the Hamiltonian takes the following general form,

$$\begin{aligned} H &= -\frac{\hbar^2}{2m} [\nabla^2 + (\alpha M^2 - \beta K)] \\ &= -\frac{\hbar^2}{2m} \left(\frac{1}{a^2 + u^2} \frac{\partial^2}{\partial v^2} + \frac{u}{(a^2 + u^2)} \frac{\partial}{\partial u} + \frac{\partial^2}{\partial u^2} + \beta \frac{a^2}{(a^2 + u^2)^2} \right), \end{aligned} \quad (87)$$

where,

$$M = 0, \quad K = -\frac{a^2}{(a^2 + u^2)^2}. \quad (88)$$

From the Dirac's brackets (82), the first category of the fundamental commutators between operators q^μ and p_ν are given by,

$$[q^\mu, q^\nu] = 0, [p_\mu, p_\nu] = 0, [q^\mu, p_\nu] = i\hbar\delta_\nu^\mu. \quad (89)$$

Similarly, we have the second category of fundamental commutators between q^μ and H from Eq. (84),

$$[u, H] = \frac{i\hbar}{m}p_u, \quad (90)$$

$$[v, H] = i\hbar \frac{p_v}{m(u^2 + a^2)}. \quad (91)$$

On the other, the quantum commutators (90) and (91) from Hamiltonian (87) give a definite and satisfactory form for the operator p_u and p_v , [26]

$$p_u = -i\hbar \left(\frac{\partial}{\partial u} + \frac{u}{2(a^2 + u^2)} \right), \quad (92)$$

$$p_v = -i\hbar \frac{\partial}{\partial v}. \quad (93)$$

Using these operators, we can directly calculate two quantum commutators $[p_u, H]$ and $[p_v, H]$ with quantum Hamiltonian (87), and the results are, respectively,

$$[p_u, H] = i\hbar \{p_u, H\}_D - \frac{i\hbar^3 u}{4m(a^2 + u^2)^3} [a^2(-5 + 8\beta) + u^2], \quad (94)$$

$$[p_v, H] = 0. \quad (95)$$

The second equation (95) is satisfactory, whereas the first one (94) is problematic. there is a manifest breakdown of the formal algebraic structure between $\{p_u, H\}_D$ and $[p_u, H]$ no matter what value of the parameter β is chosen.

C. Remarks

From the studies in this section, we see that the generalized Dirac's theory of second-class constraints for quantum motion on the helicoid can not be consistently established. We therefore need to invoke an extrinsic examination of the same problem, as will be done in next section.

V. DIRAC'S THEORY OF SECOND-CLASS CONSTRAINTS FOR A HELICOID AS A SUBMANIFOLD

The surface equation of the helicoid (65) in Cartesian coordinates (x, y, z) is given by,

$$f(\mathbf{x}) \equiv z - a \arctan \frac{y}{x} = 0. \quad (96)$$

In this section, we will also first give the classical mechanics for motion on the helicoid within Dirac's theory of second-class constraints, and then turn into quantum mechanics. The obtained momentum and Hamiltonian are all in agreement with those given by Schrödinger theory.

A. Classical mechanical treatment

The Lagrangian L in the Cartesian coordinate system is,

$$L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \lambda f(\mathbf{x}). \quad (97)$$

The generalized momentum p_i ($i = x, y, z$) and p_λ canonically conjugate to variables x_i ($x_1 = x, x_2 = y, x_3 = z$), and λ , are given by, respectively,

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i, (i = 1, 2, 3), \quad (98)$$

$$p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0. \quad (99)$$

Eq. (99) represents the primary constraint,

$$\varphi_1 \equiv p_\lambda \approx 0. \quad (100)$$

By the Legendre transformation, the primary Hamiltonian H_p is,

$$H_p = \frac{1}{2m} p_i^2 + \lambda f(\mathbf{x}) + u p_\lambda. \quad (101)$$

The secondary constraints are determined by successive use of the Poisson brackets,

$$\varphi_2 \equiv \{\varphi_1, H_p\} = -(z - a \arctan \frac{y}{x}) \approx 0, \quad (102)$$

$$\varphi_3 \equiv \{\varphi_2, H_p\} = \frac{a(p_y x - p_x y) - p_z (x^2 + y^2)}{m(x^2 + y^2)} \approx 0, \quad (103)$$

$$\varphi_4 \equiv \{\varphi_3, H_p\} = \frac{\lambda(a^2 + x^2 + y^2)}{m(x^2 + y^2)} + \frac{2a(-p_y x + p_x y)(p_x x + p_y y)}{m^2(x^2 + y^2)^2} \approx 0. \quad (104)$$

Similarly, the Dirac bracket between two variables A and B is defined by,

$$\{A, B\}_D = \{A, B\} - \{A, \varphi_\xi\} D_{\xi\zeta}^{-1} \{\varphi_\zeta, B\}, \quad (105)$$

where the 4×4 matrix $D \equiv \{D_{\xi\zeta}\}$ whose elements are defined by $D_{\xi\zeta} \equiv \{\varphi_\xi, \varphi_\zeta\}$ with $\xi, \zeta = 1, 2, 3, 4$ from Eqs. (100) and (102)-(104). The inverse matrix D^{-1} is easily carried out,

$$D^{-1} = \begin{pmatrix} 0 & D_{12}^{-1} & D_{13}^{-1} & D_{14}^{-1} \\ -D_{12}^{-1} & 0 & D_{23}^{-1} & 0 \\ -D_{13}^{-1} & -D_{23}^{-1} & 0 & 0 \\ -D_{14}^{-1} & 0 & 0 & 0 \end{pmatrix}, \quad (106)$$

where,

$$D_{12}^{-1} = \frac{4a^2(p_y x - p_x y)^2}{m(x^2 + y^2)(a^2 + x^2 + y^2)^2}, \quad (107)$$

$$D_{13}^{-1} = \frac{2a^2(p_x x + p_y y)}{(a^2 + x^2 + y^2)^2}, \quad (108)$$

$$D_{14}^{-1} = -D_{23}^{-1} = \frac{m(x^2 + y^2)}{a^2 + x^2 + y^2}. \quad (109)$$

Then primary Hamiltonian H_p assumes its usual one: $H_p \rightarrow H$,

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m}. \quad (110)$$

All fundamental Dirac's brackets are as follows,

$$\{x_i, x_j\}_D = 0 \quad (111)$$

$$\{x_i, p_j\}_D = \delta_{ij} - \frac{a^2}{(x^2 + y^2)(a^2 + x^2 + y^2)} \chi_i \chi_j, \quad (112)$$

$$\begin{aligned} \{p_i, p_j\}_D = & -\frac{a^2}{(x^2 + y^2)(a^2 + x^2 + y^2)} \left\{ \chi_i \left(\delta_{1j} p_y - \delta_{2j} p_x + \delta_{3j} \frac{2(x p_x + y p_y)}{a} \right) \right. \\ & \left. - \chi_j \left(\delta_{1i} p_y - \delta_{2i} p_x + \delta_{3i} \frac{2(x p_x + y p_y)}{a} \right) \right\} \end{aligned} \quad (113)$$

$$\{x_i, H\}_D = \frac{p_i}{m} = \dot{x}_i, \quad (114)$$

$$\{p_i, H\}_D = -\frac{2a(p_x x + p_y y)}{(x^2 + y^2)(a^2 + x^2 + y^2)} p_z \chi_i = \dot{p}_i, \quad (115)$$

where $\chi_i = \delta_{1i} y - \delta_{2i} x + \delta_{3i} (x^2 + y^2) / a$.

B. Quantum mechanical treatment

Now let us turn to quantum mechanics. The first category of the fundamental commutators between operators x_i and p_i are, by quantization of (111)-(113),

$$[x_i, x_j] = 0, \quad [x_i, p_j] = i\hbar \left(\delta_{ij} - \frac{a^2}{(x^2 + y^2)(a^2 + x^2 + y^2)} \chi_i \chi_j \right), \quad (116)$$

$$[p_i, p_j] = i\hbar \{p_i, p_j\}_D. \quad (117)$$

However, with the help of the second category of the fundamental commutators as $[x_i, H]$ and $[p_i, H]$, we immediately find that the momentum from following commutator,

$$[x_i, H] = i\hbar \frac{p_i}{m}. \quad (118)$$

They are, respectively, irrespective of the form of the geometric potential,

$$p_x = -i\hbar \left(\cos v \frac{\partial}{\partial u} - \frac{u \sin v}{a^2 + u^2} \frac{\partial}{\partial v} \right), \quad (119)$$

$$p_y = -i\hbar \left(\sin v \frac{\partial}{\partial u} + \frac{u \cos v}{a^2 + u^2} \frac{\partial}{\partial v} \right), \quad (120)$$

$$p_z = -i\hbar \frac{a}{a^2 + u^2} \frac{\partial}{\partial v}. \quad (121)$$

They are nothing but the geometric momentum (1) on the helicoid.

As to the form of quantum Hamiltonian, we also assume the general form (87). For the quantum commutators of the operators p_x, p_y, p_z and H , we must have from (115),

$$[p_i, H] = i\hbar \frac{1}{2} (F_i + F_i^\dagger) \quad (122)$$

where

$$F_i = -\frac{2a(p_x x + p_y y)}{(x^2 + y^2)(a^2 + x^2 + y^2)} p_z \chi_i, \quad \chi_i = \delta_{1i} y - \delta_{2i} x + \delta_{3i} \frac{(x^2 + y^2)}{a}. \quad (123)$$

We can easily show that the geometric potential with $\beta = 1$ is compatible with the SCQ. For instance, we have,

$$\begin{aligned} [p_i, H] = i\hbar \left\{ -\alpha_1 \left[x g_i \frac{1}{2} (p_x p_z + p_z p_x) + (x \rightleftharpoons y, p_x \rightarrow p_y) \right] \right. \\ \left. - \alpha_2 \left[\frac{1}{2} (p_x p_z + p_z p_x) x g_i + (x \rightleftharpoons y, p_x \rightarrow p_y) \right] \right. \\ \left. - \alpha_3 \left[\frac{1}{2} (p_x x g_i p_z + p_z x g_i p_x) + (x \rightleftharpoons y, p_x \rightarrow p_y) \right] \right\} \end{aligned} \quad (124)$$

where α_k , ($k = 1, 2, 3$) are three real parameters satisfying $\sum \alpha_k = 1$, $g_i = 2a\chi_i/(m(x^2 + y^2)(a^2 + x^2 + y^2))$. In comparison of both sides of the this equation, we find β and three real parameters α_k are freely to be specified,

$$\alpha_1 = \alpha_2 = \frac{1}{2} - \frac{\alpha_3}{2}, \beta = -\frac{3}{4}(1 - \alpha_3). \quad (125)$$

when the free parameter α_3 is defined as $-1/3$, we can find $\beta = 1$ in (87), the geometric potential given by Dirac formalism matches with Schrödinger theory.

C. Remarks

An examination of the motion on helicoid as a submanifold problem in Dirac's theory of second-class constraints ensures a self-consistent description.

VI. DISCUSSIONS AND CONCLUSIONS

A revisit of relation between Cartesian coordinates and canonical quantization scheme for quantum motions on catenoid and helicoid is done extensively. Within the intrinsic geometry, the generalized momenta can all be obtained within the SCQ, but the geometric potentials can hardly be self-consistently formulated into the theory in general. In contrast, canonical quantization in three-dimensional flat space in which the surface is embedded is successful, and we have not only the geometric momenta, but also the geometric potentials that can be after all obtained with help of the rearrangement of the operator-ordering. We can safely conclude that intrinsic geometry does not in general offer a framework for quantum mechanics to be satisfactorily formulated.

This study is compatible with the Dirac's insightful remark, implying that the preferable Cartesian coordinate system has fundamental importance in passing from the classical Hamiltonian to its quantum mechanical form. For particles move on the curved manifold, we have to avoid the intrinsic geometry, and search for higher-dimensional flat space the manifold is embedded.

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